Sample worksheets and exams

In this appendix, I include some sample worksheets, intended to provide a glimpse at the kinds of questions students were exploring on a day-to-day basis. I've also included the exams I gave during the course as an illustration of the topics I felt were important and what I felt the students were prepared to do.

Math 761-Spring 96
Group problems for Monday, February 12 (and beyond)
Definition of group (and its consequences)

As we defined in class, a set S, along with an operation *diamond*, is a group if the following properties are satisfied:

1. For all a and b in S, $a \diamond b$ is in S. We then say that S is closed under the operation $\diamond$.

2. For all a, b, and c in S, $(a \diamond b) \diamond c = a \diamond (b \diamond c)$. We then say that $\diamond$ is associative on S.

3. There exists an element e in S so that $e \diamond a = a \diamond e = a$ for each a in S. We then say that e is the identity element of S.

4. For each a in S, there is an element a^{-1} in S so that $a \diamond a^{-1} = a^{-1} \diamond a = e$ (where e is an identity element of S). We say that a^{-1} is the inverse of a in S.

Problems for group discussion:

1. Determine whether the subset $\{2, 4, 6, 8\}$ of $\mathbb{Z}_{10}$ is a group under multiplication (in $\mathbb{Z}_{10}$).

   In earlier problems with operation tables, we noticed that in some rows and columns of some tables, not all elements of the set occurred. For example, consider the multiplication table for $\mathbb{Z}_{10}$. Several of the rows and columns have repeated elements, so some elements have to be left out; for example, 1 is not in the 2 row.

2. In an earlier problem, you were asked to determine whether $ab = ac$ implies $b = c$ for a variety of sets and associated operations. Is this *cancellation law* true for groups? That is, if S is a group under the operation $\diamond$ and $a \diamond b = a \diamond c$, then is it necessarily true that $b = c$? In other words, is it possible to cancel the a from both sides of the equation?

3. The above version of the cancellation law is often called the *left* cancellation law, since the a is canceled on the left. Determine whether the *right* cancellation law is true for groups.

4. What do the right and left cancellation laws imply about whether the rows and columns in a groups operation table have any repeated elements?

5. Show that each row and column in a groups operation table contains every group element exactly once. For example, to show that the element c occurs in the a row, we need to show that the equation $a \diamond x = c$ has a solution in the group. Show that this is true and explain why it shows that c is in the a row, and show that c also occurs in the a column.

6. Suppose that $(S, \diamond)$ is a group with identity e. If $a \diamond x = a$ for some a and x in S, what can you conclude about x? What if $x \diamond a = a$? That is, what kinds of elements can act like an identity in a group?

7. How many inverses can an element in a group have? That is, if a is an element of a group S with operation $\diamond$ and identity e, and if $x \diamond a = e$ or $a \diamond x = e$, what can you conclude about x ?

8. If S is a group with operation $\diamond$ and identity e, and a and b are elements of S so that $a^{-1} = b^{-1}$, what can you conclude about a and b?

   Additional terminology:

   Let S be a group under the operation $\diamond$. If $a \diamond b = b \diamond a$ for all a and b in S, then we say that $\diamond$ is *commutative* on S and that S is an *abelian* group. If S is not abelian, we say that S is nonabelian.

9. Show that the mixed cancellation law is true for abelian groups. That is, prove that if the group S, with operation $\diamond$, is abelian and if $a \diamond b = c \diamond a$, then $a = c$.

Math 761 – more problems – March 27, 1996
Due during the next week

If G is a group, we define **the center of G** to be the set $\{x \in G \mid gx = xg \text{ for all } g \in G\}$. Typically, this set is denoted $Z(G)$ (the Z stands for zentrum, which is German for center). Colloquially, the center of G consists of the elements of G which commute with every element of G.

1. Find $Z(D_3)$, $Z(D_4)$, and $Z(\mathbb{Z}_8)$.

2. Determine whether each of the centers above are subgroups of their corresponding groups.

3. Prove that the set $\{e\}$, consisting only of the identity of the group G, is a subgroup of G. It is known as the trivial subgroup of G.

4. Are there any other subgroups of a group G which consist of exactly one element?

5. Let G be a group and call its center Z.

   (a) Carefully interpret the statement "c is in the center of G."

   (b) If e is the identity of G, is e in Z?

   (c) If x is in the center of G, determine whether x^{-1} is in the center, too.

   (d) Show that Z is closed under G's operation.

   (e) What may we now conclude about the center of a group?

6. Suppose a is an element of a group G. The **centralizer of a in G** is the set $\{x \in G \mid ax = xa\}$, and is denoted $C(a)$ or $C_G(a)$.

   (a) Let $G = D_4$ and let a be the element $(13)(24)$, expressed in "cycle notation" (that is, $a(1) = 3$, $a(2) = 4$, $a(3) = 1$, and $a(4) = 2$). Find the centralizer of a in D_4 and determine whether it is a subgroup of D_4.

   (b) Let G be an arbitrary group containing the element a and determine whether $C_G(a)$ is always a subgroup of G.

7. Express D_3 and D_4 using cycle notation. You might find it useful to use the handout you were given in class on Monday.

8. Express the elements of S_4, the permutations on the set $\{1, 2, 3, 4\}$ using cycle notation. It might help to examine the possibilities in terms of their corresponding "cycle structure". How many "1-cycles" are there? 2-cycles? 3-cycles? 4-cycles? Are their other possible "structures"? How many elements will S_4 have?

9. Using *Exploring Small Groups*, find the group table for A_4 and determine whether the subset $\{g2 \mid g \in A_4\}$ is a subgroup of A_4.

Exam 1 – in class portion

1. (a) Explain how we know that there are no integer solutions to $3x \equiv 8(\text{mod } 90)$.
   (b) Find all integers x so that $0 \leq x < 91$ and $3x \equiv 8 \pmod{92}$. (you may find it helpful to notice that $3 \cdot 31 \equiv 1 \pmod{92}$.)
2. Write complete definitions of the following phrases. That is, carefully describe exactly what each phrase means.
   (a) e is an identity of the set S under the operation $\diamond$.
   (b) The operation $\diamond$ is associative on the set S.
   (c) The operation $\diamond$ is commutative on the set S.
   (d) In the group G, $x^{-1} = y$.
   (e) The set S is a group under the operation $*$.
3. As you might expect, $\frac{1}{2}\mathbb{Z}$ is defined to be the set $\{\frac{1}{2}z \mid z \in \mathbb{Z}\}$.
   (a) Confirm that $\frac{1}{2}\mathbb{Z}$ is a group under addition.
   (b) Is $\frac{1}{2}\mathbb{Z}$ a group under multiplication? Explain.
4. (a) Construct the multiplication table for $\{4, 8, 12, 16\}$ in $\mathbb{Z}_{20}$.
   (b) Determine whether $\{4, 8, 12, 16\}$ is a group under multiplication (mod 20).
5. The following operation table is not a group table. Which properties fail? Explain. You may use the fact that you know that this isn't a group table in your explanation .

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	d	e	f	a
c	c	d	a	f	b	e
d	d	e	f	a	c	b
e	e	f	b	c	a	d
f	f	a	e	b	d	c

Exam 1 – takehome portion

1. The set $\{e, a, b, c, d\}$ is a group under the operation which is given in the table below. Unfortunately, I only had time to copy down part of the table. Fill out the rest of the table and check to see that we get a group. Explain how you know that your choice for each entry is the only possibility. Then use Exploring Small Groups to confirm that you have, indeed, created a group table.

$*$	e	a	b	c	d
e	e				
a		b			e
b		c	d	e	
c		d		a	b
d					

2. Fill out all possible operation tables which make the set $\{e, a, b, c\}$ a group. You may assume that e is the identity of each group. In the process, be sure to explain your decisions and prove that the set is a group under each of the specified operations. You may use the computer to check the associative property, but you must justify all of the other properties by hand. You don't have to provide a formula for the operation, but explain each decision you make while filling out the table. For example, explain how we know that the first row and column is the same for each possible table. What are the possibilities for a^2? Are there any that don't work (that is, that won't allow the set to be a group)? In a particular table, once you've chosen a^2, does this force some (or all) of the other values in the table? Why can't we have $bc = b$? Is it possible to fill out the table so that the set is a nonabelian group?

3. Recall that the funny addition operation $*$ on the set $\mathbb{R}$ of real numbers is defined by $a * b = a + b + ab$. Show that $*$ satisfies the associative property .

4. Prove of disprove: If x, y, and z are elements of a group G, then $(xy^{z-1})^{-1} = x^{-1}y^{-1}z$.

5. Prove or disprove: If x, y, and z are elements of a group G, then $(xyz^{-1})^{-1} = zy^{-1}x^{-1}$.

Exam 2 – takehome

1. Are the following statements true, or false? Prove or find a counterexample.

   (a) If G is an abelian group, then the set $\{g \in G \mid g^2 = e\}$ is a subgroup of G.

   (b) If G is a group, then the set $\{g \in G \mid g^2 = e\}$ is a subgroup of G.

2. Find the subgroup generated by the given element(s) in the specified group.

   (a) The subgroup of S_4 generated by (134).

   (b) The subgroup of S_5 generated by (124)(35).

   (c) The subgroup of S_4 generated by (14) and (124). Additional question: Is there an element a in S_4 so that the subgroup you just found is generated by a ? Explain.

3. Let the function $f : \mathbb{Z} \to \mathbb{Z}_4$ be defined by $f(x) = x \pmod 4$.

   (a) Show that f is a homomorphism. Is f one-to-one? Is f onto?

   (b) Find the kernel of f. Recall that the kernel of a homomorphism $f : G \to G'$ is the set $ker(f) = \{g \in G \mid f(g) = e'\}$, where e' is the identity of G'.

4. Suppose $f : G \to G'$ is a group homomorphism.

   (a) Prove that $ker(f)$ is a subgroup of G.

   (b) Prove that if g is an element of G, then $(f(g))^{-1} = f(g^{-1})$.

   (c) Prove that $f(G)$ is a subgroup of G. Recall that $f(G) = \{f(g) \mid g \in G\}$.

   (d) Prove that if G is an abelian group, then $f(G)$ is an abelian group.

5. Let K be the kernel of the group homomorphism $f : G \to G'$ and suppose a and b are elements of G.

   (a) Show that if b is in the set $aK = \{ak \mid k \in K\}$, then $f(b) = a$.

(b) Show that if $f(a) = f(b)$, then there exists an element k in K so that $b = ak$ by following the steps below.

   i) First, explain how we know that there is a k in G so that b=ak.

   ii) Now show that the k you found in part i) is in K.

(c) Explain why we may now conclude that $aK = \{x \in G \mid f(x) = f(a)\}$, and therefore $f(a) = f(b)$ if and only if $aK = bK$.

6. Prove that if f is a group homomorphism, then f is one-to-one if and only if $ker(f) = e$.

Math 761
Final exam
Friday, May 10

1. Provide complete definitions for the following terms and phrases. Be sure to explain any otherwise undefined terms.

   (a) associative

   (b) commutative

   (c) group

   (d) subgroup

   (e) homomorphism

   (f) isomorphism

   (g) the order of the element a in the group G

2. Carefully define the following groups, being sure to include the operation under which these sets are groups.

   (a) $\mathbb{Z}_n$

   (b) $\mathbb{U}_n$

3. Describe the following sets in words AND using set notation.

   (a) the kernel of the group homomorphism $f : G \to G'$.

   (b) $A + B$ (where A and B are subsets of the integers)

   (c) $A * B$ (where A and B are subsets of the group G, under the operation $*$)

4. Let α be the permutation of $\{1, 2, 3, 4\}$ that sends 1 to 2, 2 to 3, 3 to 1, and 4 to 4, and let β be the permutation that sends 1 to 4, 2 to 3, 3 to 2, and 4 to 1. Find the order of $\alpha\beta^2\alpha^{-1}$.

5. Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be defined by $f(z) = 2z$.

   (a) Determine whether f is a homomorphism.

   (b) Determine whether f is one-to-one.

   (c) Determine whether f is onto.

   (d) Determine whether $\mathbb{Z}$ is isomorphic to $2\mathbb{Z}$.

6. Recall that if g is an element of G, then the cyclic subgroup of G generated by g is defined to be the set $\langle g \rangle = \{g^n \mid n \epsilon \mathbb{Z}\}$.

   (a) How does the order of $\langle g \rangle$ compare to the order of g? Provide a brief explanation.

   (b) If G has finite order and g is an element of G, can g have infinite order? Explain.

7. Recall that Lagrange's Theorem states that if H is a subgroup of a <u>finite</u> group G, then the order of H is a divisor of the order of G. With this in mind, finish Lagrange's Theorem for elements:
   If g is an element of finite of a finite group G, then the order of g ____________________________.
   Provide a brief explanation of why this Theorem is true.

8. Let G be a group and define the set S as follows: $S = \{x \epsilon G \mid x^4 = x\}$

   (a) Carefully state what it means for a to be in S.

   (b) Carefully state what it means for b to <u>not</u> be in S.

   (c) Suppose $G = D_3$ and S is defined as above.

      i. Is (123) in S?

      ii. Is (13) in S?

   (d) Now let G be an arbitrary group. Is S a subgroup of G? If so, prove it. If not, what goes wrong? Under what circumstances will S be a subgroup of G? Prove that, under these circumstances, S is a subgroup of G.

9. Find the cyclic subgroup of $\mathbb{Z}_{15}$ generated by 5.

   (a) List the left cosets of H in $\mathbb{Z}_{15}$, where H is the subgroup you found above.

   (b) Is H a <u>normal</u> subgroup of $\mathbb{Z}_{15}$? Explain.

   (c) Provide the table for the quotient group $\mathbb{Z}_{15}/H$ below.

10. (a) Find the left cosets of $\{(1), (25)\}$ in the group $G = \{(1), (245), (254), (24), (25), (45)\}$.

   (b) Find the right cosets of $\{(1), (13)\}$ in G.

   (c) Compute $(12)H(123)H$.

   (d) Is H a normal subgroup of G? Explain.